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BOOLEAN POWERS OF ABELIAN GROUPS

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Let A be an abelian group and B a complete Boolean algebra (cBa). The Boolean power $A^{(B)}$ consists of all f such that

$$\bigvee_{u\in A} f(u)=1 \quad \text{and} \quad f(u)\wedge f(v)=0 \quad \text{for } u\neq v,$$

where $(f+g)(u)=\bigvee_{u=v+w} f(v)\wedge f(w)$. Since $\mathbb{Z}$ is countable, $\mathbb{Z}^{(B)}$ can be defined for any countably complete Boolean algebra (ccBa) B where $\mathbb{Z}$ is the group of the integers. This kind of group was first (1962) studied by Balcerzyk [1]. However, it seems that not much attention was paid to such groups for a rather long period. Under the point of view in [1, Theorem 5] and [13, Proposition 1], it can be said that there has been studies about it after around 1980 [5–18, 24, 29, 30].

In the present paper we investigate cotorsion-freeness and algebraical compactness of Boolean powers. Undefined notions about abelian groups and Boolean algebras are the usual ones and can be found in [20] and [23] respectively. All groups in this paper are abelian groups.

1. Cotorsion-free groups

A group A is cotorsion-free, if A does not include a nonzero cotorsion subgroup. In other words, A is torsion-free and reduced and does not contain a copy of the group of p-adic integers J_p for any prime p. It is also known that A is cotorsion-free iff $\mathrm{Hom}(\hat{\mathbb{Z}}, A)=0$, where $\hat{\mathbb{Z}}$ is the $\mathbb{Z}$-adic completion of $\mathbb{Z}$.

Proposition 1.1. *Let A be a cotorsion-free group and B a cBa. Then, the Boolean power $A^{(B)}$ is also cotorsion-free.*

Proof. To prove the contraposition, let $h:\hat{\mathbb{Z}}\to A^{(B)}$ be a nonzero homomorphism. Since $A^{(B)}$ is torsion-free and reduced, there exists $a\in A$ such that $0\neq h(1)(a)$ $(=b)$ and $a\neq 0$. Next, we show that there exists a unique $a_x\in A$ for each $x\in\hat{\mathbb{Z}}$ with $b\leqslant h(x)(a_x)$. Clearly, $b\leqslant h(n)(na)$ for $n\in\mathbb{Z}$. For each $x\in\hat{\mathbb{Z}}$, there exist $x_n\in\mathbb{Z}$ $(n<\omega)$ such that $n!\mid x-x_n$. Let $0\neq c_0, c_1\leqslant b$ so that

$c_0 \leqslant h(x)(u_0)$ and $c_1 \leqslant h(x)(u_1)$. Since $n! \mid x - x_n$, $n! \mid u_k - x_n a$ for $k = 0, 1$ and $n < \omega$ and hence $n! \mid u_0 - u_1$ $(n < \omega)$. This implies $u_0 = u_1$, because A is reduced and torsion-free. Now, let $h^*(x) = a_x$, then h^* is a nonzero homomorphism from $\hat{\mathbb{Z}}$ to A. □

Let $C_\kappa (=RO(^\omega\kappa))$ be the cBa consisting of all regular open subsets of a topological space $^\omega\kappa$, where κ is discrete and $^\omega\kappa$ is endowed with the product topology. It is well known that every Boolean algebra is completely embeddable in C_k for some κ [25]. Therefore, as we noted in [16] a group A is $\aleph_1$-free iff A is isomorphic to a subgroup of $\mathbb{Z}^{(C_\kappa)}$ for some κ. Here, we show that the groups $\mathbb{Z}^{(C_\kappa)}$ can also be used to characterize cotorsion-free groups, which is on the line of [30, 6]. The proof of the next lemma can be found in [22]. However, since the fact is fundamental and the proof is short, we include it for the reader's convenience.

Lemma 1.2 [22]. *Let A be a cotorsion-free group and h be a homomorphism from $\mathbb{Z}^\omega$ to A. Then, the set $\{m < \omega : h(\mathbf{e}_m) = a\}$ is finite for each nonzero $a \in A$, where $\mathbf{e}_m(n) = \delta_{mn}$ for $m, n < \omega$.*

Proof. Suppose that $\{m < \omega : h(\mathbf{e}_m) = a\}$ is infinite for some nonzero $a \in A$. Without any loss of generality we may assume $h(\mathbf{e}_m) = a$ for all $m < \omega$. Any element $x \in \hat{\mathbb{Z}}$ can be written as $x = \sum^*_{n<\omega} x_n$, where $x_n \in \mathbb{Z}$, $n! \mid x_n$ $(n < \omega)$ and the infinite sum is considered under the $\mathbb{Z}$-adic topology. Define $h^*(x) = h(\sum_{n<\omega} x_n \mathbf{e}_n)$, where $\sum_{n<\omega} x_n \mathbf{e}_n$ is the function whose value of n is x_n. Since A is reduced and torsion-free, h^* is well-defined and a nonzero homomorphism, which is a contradiction. □

We refer to [14] for the definition of a quasi-sheaf over a ccBa. Boolean powers of abelian groups over ccBa's have quasi-sheaf structures. Though we state the next lemma for quasi-sheaves, the reader may consider $S^\wedge$ as a Boolean power X^B in the lemma as far as the other parts of this paper concern.

Lemma 1.3. *Let B be a ccBa, (S, ρ) a quasi-sheaf over B and A a cotorsion-free group of cardinality less than κ. For a homomorphism $h : S^\wedge \to A$, let $I_h = \{b : \rho^1_{\neg b}(x) = 0$ implies $h(x) = 0$ for every $x \in S^\wedge\}$. Then, I_h is a countably complete κ-saturated ideal.*

Proof. I_h is an ideal, clearly. Let $\{b_n\, n < \omega\}$ be a pairwise disjoint subfamily of I_h and suppose that $\bigvee_{n<\omega} b_n$ $(= c)$ does not belong to I_h. Then, there exists $x \in S^\wedge$ such that $\rho^1_{\neg c}(x) = 0$ and $h(x) \neq 0$. Let x_n be the element such that $\rho^1_{b_i}(x_n) = 0$ for $i < n$ and $\rho^1_{c_n}(x_n) = \rho^1_{c_n}(x)$, where $c_n = \bigvee_{m \geqslant n} b_m$. Define $\varphi : \mathbb{Z}^\omega \to S^\wedge$ by $\varphi(\sum_{n<\omega} a_n \mathbf{e}_n) = \sum_{n<\omega} a_n x_n$. Since $\rho^1_{b_m}(x_n) = 0$ for almost all n, φ is a well-defined homomorphism. $\rho^1_{c_n}(x - x_n) = 0$ and $\neg c_n \in I_h$ and hence $h\varphi(\mathbf{e}_n) = h(x_n) = h(x) \neq 0$

for all $n<\omega$, which contradicts Lemma 1.2. Next, we show that I_h is κ-saturated. Suppose not. Then, there exists a family $\{b_\alpha: \alpha<\kappa\}$ such that $b_\alpha \notin I_h$ and $b_\alpha \wedge b_\beta \in I_h$ for $\alpha \neq \beta$, and also $\{x_\alpha: \alpha<\kappa\} \subset S^\wedge$ such that $h(x_\alpha) \neq 0$ and $\rho^1_{\neg b_\alpha}(x_\alpha)=0$. Since the cardinality of A is less than κ, there exist $\{\alpha_n: n<\omega\} \subset \kappa$ such that $\alpha_m \neq \alpha_n$ for $m \neq n$ and $h(x_m)=h(x_n) \neq 0$ for all m, n. Since $b_\alpha \wedge b_\beta \in I_h$ for $\alpha \neq \beta$, we get $y_n \in S^\wedge$ $(n<\omega)$ and a pairwise disjoint family $\{c_n: n<\omega\}$ such that $\rho^1_{\neg c_n}(y_n)=0$ and $h(y_m)=h(y_n) \neq 0$ for all $m, n<\omega$. Define $\psi: \mathbb{Z}^\omega \to S^\wedge$ by $\psi(\sum_{n<\omega} a_n \mathbf{e}_n)=\sum_{n<\omega} a_n y_n$. Then, $h\psi(\mathbf{e}_m)=h\psi(\mathbf{e}_n) \neq 0$ for all $m, n<\omega$, which is a contradiction. □

Theorem 1.4. *The following propositions are equivalent for each group A:*

(1) *A is cotorsion-free.*

(2) $\mathrm{Hom}(\mathbb{Z}^{(C_\kappa)}, A)=0$ *for some κ.*

(3) *There exists a cardinal κ such that* $\mathrm{Hom}(\mathbb{Z}^{(C_\lambda)}, A)=0$ *for any $\lambda \geqslant \kappa$.*

Proof. First we show that there exists no countably complete κ-saturated ideal I of C_κ which is not trivial, i.e. $I \neq C_\kappa$. There exists a system $\{b_{m\alpha}: m<\omega,\ \alpha<\kappa\}$ such that $b_{m\alpha} \wedge b_{m\beta}=0$ for $\alpha \neq \beta$ and $\bigvee_{m<\omega} b_{mf(m)}=1$ for all $f \in {}^\omega\kappa$, i.e. $b_{m\alpha}=\{x \in {}^\omega\kappa: x(m)=\alpha\}$. Suppose that I is a countably complete κ-saturated ideal of C_κ. Then, there exists $\alpha>\kappa$ such that $b_{m\alpha} \in I$ and hence let $f(m)$ be such an α. By the countable completeness of I, $1=\bigvee_{m<\omega} b_{mf(m)} \in I$ and $I=C_\kappa$, which is a contradiction.

As we noted in [14], there is a sheaf S over C_κ such that $S^\wedge=\mathbb{Z}^{(C_\kappa)}$. Now, the implication (1)→(3) is clear from Lemma 1.3. It is enough to show the implication (2)→(1). If A includes a nonzero torsion group, then we may assume that the cyclic p-group is a subgroup of A for some p. Of course, $\mathbb{Z}^{(C_\kappa)}/p\mathbb{Z}^{(C_\kappa)} \neq 0$ and hence we get a nonzero homomorphism $h: \mathbb{Z}^{(C_\kappa)} \to A$. The case that A is not reduced is clear, since $\mathbb{Z}^{(C_\kappa)}$ is torsion-free. In the other case, A must include a subgroup isomorphic to J_p for some p. Since J_p is algebraically compact and $\mathbb{Z}^{(C_\kappa)}$ includes a nonzero free pure subgroup, there exists a nonzero homomorphism $h: \mathbb{Z}^{(C_\kappa)} \to A$. □

We refer the reader to [20, §85] for information about type. Let A be a group of homogeneous type. Then, it is easy to see that the Boolean power $A^{(B)}$ also has the same homogeneous type. Therefore, we get the next corollary from Proposition 1.1 and Theorem 1.4. (See also [8] for this topic.)

Corollary 1.5 (Göbel and Shelah [21]). *Let A be a cotorsion-free group. Then, there exists a cotorsion-free group X such that* $\mathrm{Hom}(X, A)=0$. *Moreover, we can take X with a given homogeneous idempotent type.*

Let A be a group of cardinality less than the least measurable cardinal μ. If F is a countably complete ultrafilter on I, then F is μ-complete as is well known and hence $A^I/F \simeq A$ and $\mathrm{Hom}(A^I/F, A) \neq 0$, where A^I/F is an ultraproduct [19]. Here, we show that the condition of the cardinality of A is necessary.

Theorem 1.6. *Let μ be the least measurable cardinal. These exists an abelian group A of cardinality μ such that* $\mathrm{Hom}(A^I/F, A) = 0$ *for any set I and any non-μ^+-complete ultrafilter F on I. Consequently, if there exists only one measurable cardinal, which is μ, then* $\mathrm{Hom}(A^I/F, A) = 0$ *for any set I and any non-principal ultrafilter F on I.*

Proof. Let D_μ be the countably complete subalgebra of C_μ which is generated by $\{b_{n\alpha}: n < \omega,\ \alpha < \mu\}$, where the $b_{n\alpha}$ are defined in the proof of Theorem 1.4. Let $A = \mathbb{Z}^{(D_\mu)}$, then $|D_\mu| = \mu$, $|A| = \mu^\omega = \mu$.

If F is not countably complete, A^I/F is algebraically compact as is well known [19, Lemma 1.8], hence we get the conclusion. Otherwise, we use an elementary embedding of the universe and hence refer the reader to [26] for it. Let j be the elementary embedding $j: V \to M$, where M is the transitive collapse of V^I/F. Then,

$$A^I/F \simeq j(A) = (\mathbb{Z}^{(D_{j(\mu)})})^M = \mathbb{Z}^{(D^M_{j(\mu)})}.$$

Now, $D^M_{j(\mu)}$ is a countably complete Boolean algebra and $\mathbb{Z}^{(D^M_{j(\mu)})}$ has a quasi-sheaf structure. Suppose that $\mathrm{Hom}(A^I/F, A) \neq 0$, then there exists a nontrivial μ^+-saturated countably complete ideal of $D^M_{j(\mu)}$ by Lemma 1.3. However, $D^M_{j(\mu)}$ has a system $\{b_{n\alpha}: m < \omega,\ \alpha < j(\mu)\}$ such that

$$\text{“}b_{n\alpha} \wedge b_{n\beta} = 0\ (\alpha \neq \beta) \quad \text{and} \quad \bigvee_{n<\omega} b_{nf(n)} = 1 \quad \text{for all } f \in {}^\omega j(u)\text{”}$$

holds in M. Since the statement also holds in V and $|j(\mu)| \geqslant \mu^+$ [26, Theorem 1.8], we get a contradiction as in the proof of Theorem 1.4. □

Remark. Let $V^{(B)}$ be a Boolean-valued model [4]. If A is cotorsion-free, then $\|A^\vee$ is cotorsion-free$\|^{(B)} = 1$ by Proposition 1.1 Actually, Proposition 1.1 is a little bit stronger, i.e. $\mathrm{Hom}(J_p^\vee, A^\vee) = 0$ in $V^{(B)}$ for any prime p. A group A is slender if A is cotorsion-free and does not contain a copy of $\mathbb{Z}^\omega$. We do not know whether $A^\vee$ is slender in $V^{(B)}$ for a slender group A in general. We have introduced a stronger notion 'primitive slenderness' in [18] and used the fact that the above proposition holds for primitive slenderness. The affirmative answer to this question would improve some results in [18, §5].

2. Algebraically compact groups

A couple of equivalent statements for algebraical compactness of abelian groups are known [20, Theorem 38.1]. Among them we adopt the following due to [7]:

(1) In $n! \mid a_{n+1} - a_n$ $(n < \omega)$, then there exists $a \in A$ such that $n! \mid a - a_n$ $(n < \omega)$.

(2) $U(A) = U(U(A))$, where $U(A) = \bigcap_{n<\omega} nA$.

Algebraical compactness of Boolean powers was first studied by Balcerzyk [1]. We proved that $J_p^{(B)}$ is algebraically compact iff B is $(\omega, 2)$-distributive for a cBa B [14, Proposition 4]. This holds also for a $(2^{\aleph_0})^+$-complete Ba by the same proof. According to the next method we must confine ourselves to cBa's, but it will clarify the behavior of algebraical compactness of Boolean powers. In the following we assume that the reader is familiar with Boolean-valued models $V^{(B)}$ [4, 27]. We assume that $V^{(B)}$ is separated, i.e. $\|x = y\|^{(B)} = 1$ implies $x = y$. For $x \in V^{(B)}$,

$$x^\wedge = \{y \in V^{(B)}: \|y \in x\|^{(B)} = 1\}.$$

(We abbreviate "$\|\cdots\|^{(B)}$" by "$\|\cdots\|$".) An element x of $A^{(B)}$ can be written as $\sum_{\lambda\in\Lambda} a_\lambda 1_{b_\lambda}$, where $a_\lambda \in A$ $(\lambda \in \Lambda)$ and $\{b_\lambda: \lambda \in \Lambda\}$ is a pairwise disjoint family, i.e. $b_\lambda \leq \|x = (a_\lambda)^\vee\|$ $(\lambda \in \Lambda)$ and $\neg\bigvee_{\lambda\in\Lambda} b_\lambda \leq \|x = 0\|$. This notation is common with that of [10]. The fundamental fact is that the Boolean power $A^{(B)}$ is isomorphic to $(A^\vee)^\wedge$. For any group X in $V^{(B)}$, i.e. $\|X$ is a group$\| = 1$, there exists a sheaf (S, ρ) over B naturally corresponding to X [14].

Lemma 2.1. *Let X be a group in $V^{(B)}$. Then, $X^\wedge$ is algebraically compact iff X is algebraically compact in $V^{(B)}$.*

Proof. By the so-called maximal principle, any sequence x_n $(n \in \mathbb{N})$ of elements of X in $V^{(B)}$ can be considered as a sequence of elements of $X^\wedge$. Moreover, for $n \in \mathbb{N}$ and $x \in X^\wedge$, $n \mid x$ iff $n^\vee \mid x$ in $V^{(B)}$. Hence, it is easy to complete the proof. □

Let X be a subgroup of J_p which contains 1. Then, X is algebraically compact iff $X = J_p$. Now, $J_p^{(B)}$ is algebraically compact iff $J_p^\vee$ is algebraically compact in $V^{(B)}$ by Lemma 2.1. The latter holds iff $J_p^\vee = J_p$ in $V^{(B)}$. Since an element of J_p is uniquely represented by a countable sequence of $0, 1, \cdots, p-1$, $J_p^\vee = J_p$ in $V^{(B)}$ iff B is $(\omega, 2)$-distributive, which is a Boolean model-theoretic proof of [14, Proposition 4].

We investigate Boolean powers of other algebraical compact groups in connection with (ω, κ)-distributivity. Let $\tilde{\mathbb{Z}}^\kappa = \{f \in \mathbb{Z}^\kappa: \operatorname{supp}(f)$ is countable$\}$ and $\bigoplus_\kappa \mathbb{Z} = \{f \in \mathbb{Z}^\kappa: \operatorname{supp}(f)$ is finite$\}$ as usual. It is well-known that $\tilde{\mathbb{Z}}^\omega / \bigoplus_\omega \mathbb{Z}$ $(=\mathbb{Z}^\omega / \bigoplus_\omega \mathbb{Z})$ is algebraically compact (due to Balcerzyk) and an essentially same proof shows that $\tilde{\mathbb{Z}}^\kappa / \bigoplus_\kappa \mathbb{Z}$ is also algebraically compact [20, Theorem 42.1].

Theorem 2.2. *Let $A_\kappa = \tilde{\mathbb{Z}}^\kappa / \bigoplus_\kappa \mathbb{Z}$ for an infinite cardinal κ, and B a cBa. The Boolean power $A_\kappa^{(B)}$ is algebraically compact iff B is (ω, κ)-distributive.*

Proof. Suppose that B is (ω, κ)-distributive, then B is also (ω, κ^ω)-distributive. Therefore, $(\tilde{\mathbb{Z}}^\kappa)^\vee = \tilde{\mathbb{Z}}^{\kappa^\vee}$, $(A_\kappa)^\vee = \tilde{\mathbb{Z}}^{\kappa^\vee} / \bigoplus_{\kappa^\vee} \mathbb{Z}$ in $V^{(B)}$ and hence $(A_k)^\vee$ is algebraically compact in $V^{(B)}$. By Lemma 2.1 we conclude that $A_\kappa^{(B)}$ is algebraically compact. To show the converse, suppose that B is not (ω, κ)-distributive. Then, there exists $f \in V^{(B)}$ such that

$$\|f: \omega^{\vee} \to \kappa^{\vee} \text{ is injective and } f \notin V^{\vee}\| = b \neq 0.$$

Let $\kappa = \bigcup_{\alpha<\kappa} I_\alpha$ so that $|I_\alpha| = \aleph_0$ and $I_\alpha \cap I_\beta = \emptyset$ $(\alpha \neq \beta)$. Let $C = [0, b]$ and we work in $V^{(C)}$ in the following. Define $x: \kappa^{\vee} \to \mathbb{Z}$ by $x(\gamma) = n!$ if $\gamma \in (I^{\vee})_{f(n)}$; $x(\gamma) = 0$ otherwise. Next, define x_m $(m < \omega)$ by $x_m(\gamma) = n!$ if $\gamma \in (I^{\vee})_{f(n)}$ and $n < m$; $x_m(\gamma) = 0$ otherwise. Then, x_m belongs to $(\tilde{\mathbb{Z}}^{\kappa})^{\vee}$ for all $m \in \mathbb{N}$. Moreover, $m! \mid x_{m+1} - x_m$ $(m < \omega)$ holds. Let $[\,]$: $\tilde{\mathbb{Z}}^{\kappa^{\vee}} \to \tilde{\mathbb{Z}}^{\kappa^{\vee}} / \bigoplus_{\kappa^{\vee}} \mathbb{Z}$ be the canonical quotient homomorphism. Then, $(A_\kappa)^{\vee}$ is a subgroup of $\tilde{\mathbb{Z}}^{\kappa^{\vee}} / \bigoplus_{\kappa^{\vee}} \mathbb{Z}$ and $m! \mid [x_{m+1}] - [x_m]$ $(m < \omega)$. Suppose that $(A_\kappa)^{\vee}$ is algebraically compact. There exists $y \in (\tilde{\mathbb{Z}}^{\kappa})^{\vee}$ such that $m! \mid [y] - [x_m]$ $(m < \omega)$. Then, $x(n) = \alpha$ iff $y(\gamma) \equiv n! \pmod{(n+1)!}$ for almost all $\gamma \in (I^{\vee})_\alpha$. This implies $x \in V^{\vee}$, which is a contradiction. Now, we have proved that $A_\kappa^{(C)}$ is not algebraically compact. Since $A_\kappa^{(C)}$ is a summand of $A_\kappa^{(B)}$, $A_\kappa^{(B)}$ is not algebraically compact. □

Remarks. (1) Performing a straight proof of Theorem 2.2 as in [14, Proposition 4], we can relax the completeness of B to $(\kappa^{\omega})^{+}$-completeness in the assumption of Theorem 2.2.

(2) An R-module version of Lemma 2.1 does not hold in general. We explain this fact precisely. Let $\|M$ is an S-module$\| = 1$. Then, it is rather straightforward to prove that $M^{\wedge}$ is an algebraically compact $S^{\wedge}$-module iff $\|M$ is algebraically compact$\| = 1$. (Here, M is said to be an algebraically compact S-module if any finite solvable S-coefficient system of equations is solvable in M.) Therefore, if $(A_R)^{\vee}$ is algebraically compact as an $R^{\vee}$-module in $V^{(B)}$, then $(A^{\vee})^{\wedge}$ $(\simeq A_R^{(B)})$ is algebraically compact as an R-module. The converse does not hold in general and a counterexample can be given by a module over a certain Boolean ring, which are known as typical Boolean algebras appearing in [2, 3] and which can also be found in [28]. There exist cBa's B and C such that B is (ω, ∞)-distributive, C satisfies the countable chain condition but $\|C^{\vee}$ is not complete$\|^{(B)} = 1$. By the (ω, ∞)-distributivity of B, $C^{\vee}$ is countably complete in $V^{(B)}$ and hence $C^{(B)}$ is countably complete. Since C satisfies the countable chain condition, $C^{(B)}$ is an injective C-module and so is algebraically compact. As is well known and easy to check, a Boolean ring is selfinjective iff it is complete as a Boolean algebra. It is also easy to see that a Boolean ring is an algebraically compact module over itself iff it is selfinjective. Therefore, $C^{\vee}$ is not an algebraically compact $C^{\vee}$-module in $V^{(B)}$.

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